Jordan Blocks of Richardson Classes in the Classical Groups and the Bala-Carter Theorem

W. Ethan Duckworth
334 Hill Center, Rutgers University, Piscataway NJ, 08854
duck@math.rutgers.edu

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Abstract

This paper provides new, relatively simple proofs of some important results about unipotent classes in simple linear algebraic groups. We derive the formula for the Jordan blocks of the Richardson class of a parabolic subgroup of a classical group. This result was originally due to Spaltenstein. Secondly, we derive, for good characteristic, the description of the natural partial order of unipotent classes of a classical group in terms of their Jordan blocks. This result was originally due to Gerstenhaber and Hesselink. As a consequence we obtain a proof of the Bala-Carter Theorem which holds even in certain bad characteristics (this proof requires the prior classification of unipotent classes, unlike the original proofs due to Bala, Carter and Pommerening).

Keywords: classical groups, unipotent classes, Richardson classes, partial order of unipotent classes, Jordan blocks, Bala-Carter Theorem.

1 Introduction

Let G be a connected reductive linear algebraic group. Richardson [14] made a vital contribution to the study of unipotent classes in algebraic groups by associating to each parabolic subgroup of G a unipotent class of G. This result has had surprisingly powerful implications, some of which we will discuss below. The following theorem is one version of Richardson's result (see also [4] or [19] for other proofs).

Richardson's Theorem. *Let G be a connected reductive group, P a parabolic subgroup with unipotent radical Q and Levi factor L. The following hold:*

(i) *There exists a unique unipotent G-class C such that $C \cap Q$ is open and dense in Q.*
(ii) *$C \cap Q$ forms a single P-class.*
(iii) *If $u \in C \cap Q$ then $C_G(u)^\circ = C_P(u)^\circ$, whence these centralizers have dimension* $\dim L$.
(iv) *Let Z be the center of G, let Q' be the derived subgroup of Q. Then* $\dim L/Z \geq \dim Q/Q'$.

Spaltenstein [17] studies generalizations of this result to the case where G is non-connected. We will always have G connected except when $G = \emptyset_n$.

We call C the **Richardson class** of P and we call $C \cap Q$ the **Richardson orbit** in Q.

For many questions it is of fundamental importance to be able to find the Jordan blocks of a unipotent class. The next result indicates how to do this for Richardson classes, but first we introduce some standard notation.

A **partition** of n is a sequence of natural numbers which add to n; we assume that the sequence is weakly decreasing unless indicated otherwise. We write a partition λ as $(\lambda_1, \lambda_2, \dots)$ or $(\lambda_1 \geq \lambda_2 \geq \dots)$ or $(1^{c(1)}, 2^{c(2)}, 3^{c(3)}, \dots)$ where $c(x)$ is the multiplicity of x in λ. We call λ_i a **part** of λ. The **dual** of λ is a partition of n which we write as λ^* and which has parts defined as follows: λ_i^* equals the number of parts of λ which are greater than or equal to i (i.e. λ_i^* equals the number of indices j such that $\lambda_j \geq i$).

Let $G \in \{\mathrm{SO}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{Sp}_{2n}\}$. We fix a root base Δ for G and label the nodes $\{\alpha_1, \dots, \alpha_n\}$ as in [3]. Given a parabolic subgroup P let $J \subseteq \Delta$ such that P is conjugate to the standard parabolic associated with J. If $\alpha_n \in J$ then let m be the largest integer such that such that the last m nodes $\alpha_{n-m+1}, \dots, \alpha_n$ are contained in J. Then the Levi factor L of P can be written as $L = \mathrm{GL}_{n_1} \dots \mathrm{GL}_{n_s} \mathrm{Cl}_m$ where $\mathrm{Cl}_m \in \{\mathrm{SO}_{2m}, \mathrm{SO}_{2m+1}, \mathrm{Sp}_{2m}\}$. We extend this notation also to the cases $\alpha_n \notin J$ and to $G = \mathrm{GL}_n$ by taking $\mathrm{Cl}_m = 1$ and $m = 0$. In this way each parabolic subgroup of GL_n, SO_{2n}, SO_{2n+1}, Sp_{2n} determines a partition $n = n_1 + \dots + n_s + m$. We call this the **Levi partition** and write it either as $\Lambda = (n_1, \dots, n_s) \oplus m$ or $\Lambda = (1^{c(1)}, 2^{c(2)}, \dots) \oplus m$ where $c(x)$ is the multiplicity of the part x in the n_i and the notation "$\oplus$" indicates that we have an ordered pair consisting of the partition $(n_1, \dots, n_s)$ and the number m.

Table 1: Jordan blocks of a Richardson class

		$G = \mathrm{GL}_n,\ \psi$ is the identity	
		$G = \mathrm{SO}_{2n},\ p \neq 2$	
$\psi(m)$	$=$	$2m$	
$\psi(j^{c(j)})$	$=$	$j^{2c(j)}$	if j is even or $j \leq 2m$
$\psi(j^{c(j)})$	$=$	$j+1,\ j^{2c(j)-2},\ j-1$	if j is odd and $j > 2m$
		$G = \mathrm{SO}_{2n},\ p = 2$	
$\psi(m)$	$=$	$2m$	
$\psi(j^{c(j)})$	$=$	$j^{2c(j)}$	if j is even
$\psi(j^{c(j)})$	$=$	$j+1,\ j^{2c(j)-2},\ j-1$	if j is odd and $j \leq 2m$
$\psi(j^{c(j)})$	$=$	$j+1,\ j-1$	if j is odd, $j > 2m$ and $c(j) = 1$
$\psi(j^{c(j)})$	$=$	$(j+1)^2,\ j^{2c(j)-4},\ (j-1)^2$	if j is odd, $j > 2m$ and $c(j) \geq 2$
		$G = \mathrm{SO}_{2n+1},\ p \neq 2$	
$\psi(m)$	$=$	$2m+1$	
$\psi(j^{c(j)})$	$=$	$j^{2c(j)}$	if j is odd and $j > 2m+1$
$\psi(j^{c(j)})$	$=$	$j^{2c(j)}$	if $j \leq 2m+1$
$\psi(j^{c(j)})$	$=$	$j+1,\ j^{2c(j)-2},\ j-1$	if j is even and $j > 2m+1$
		$G = \mathrm{Sp}_{2n}$	
$\psi(m)$	$=$	$2m$	
$\psi(j^{c(j)})$	$=$	$j^{2c(j)}$	if j is even or $j \geq 2m$
$\psi(j^{c(j)})$	$=$	$j+1,\ j^{2c(j)-2},\ j-1$	if j is odd and $j < 2m$

Theorem 1 ([17, II.7.4]). *Let G be one of GL_n, SO_{2n}, SO_{2n+1}, Sp_{2n} and exclude the case SO_{2n+1} if $p = 2$. Let P be a parabolic subgroup of G, let Λ be the Levi partition of P, define the map ψ as in table 1, and let λ be the partition of Jordan block sizes of the Richardson class of P. Then λ equals $\psi(\Lambda)^*$, the dual of $\psi(\Lambda)$.*

Remarks 1.1. (a) It is easy to extend this result to the case of $G = \mathrm{SO}_{2n+1}$ and $p = 2$. One applies the formula for Sp_{2n} using the same Levi partition and adds one block of size 1 to the result. (b) Spaltenstein's formulas appear rather different from those presented here (and have some minor mistakes). (Spaltenstein also determines the index ε in his notation or, equivalently, the singularity of the parts of λ. See [5] for a discussion of this notation.)

Let C_1 and C_2 be two unipotent classes of G. We define $C_1 \leq C_2$ if and only if $C_1 \subseteq \overline{C_2}$ (where $\overline{C_2}$ is the closure of C_2). This is the natural partial order on unipotent classes.

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ be two partitions. We define $\lambda \leq \mu$ if and only if for all $j \geq 1$ we have $\sum_{i=1}^{j} \lambda_i \leq \sum_{i=1}^{j} \mu_i$. This is the **dominance** partial order on partitions.

Theorem 2 ([7], [6], [17, I.2.10]). *Let $G \in \{\mathrm{GL}_n, \mathrm{O}_n, \mathrm{SO}_n, \mathrm{Sp}_n\}$. Let C_λ and C_μ be two unipotent classes in G with λ and μ the partitions consisting of the Jordan blocks of C_λ and C_μ respectively. Assume either that $p \neq 2$ if $G \neq \mathrm{GL}_n$ or that μ has no even parts with even multiplicity. Then $\lambda < \mu$ if and only if $C_\lambda < C_\mu$.*

Spaltenstein [17, I.2.10] generalizes this result to all unipotent classes in bad characteristics, but we will not discuss his generalization here.

We now introduce the necessary terminology to state the Bala-Carter Theorem which gives a parameterization of unipotent classes in simple algebraic groups.

Let G be a connected reductive algebraic group with root system Φ and root base Δ. Fix $J \subseteq \Delta$ and let P be the standard parabolic subgroup corresponding to J. Let $\beta \in \Phi$ and write $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha$. The P-height is defined to be $\mathrm{ht}_P(\beta) = \sum_{\alpha \in \Delta - J} n_\alpha$.

Let L be a Levi factor for P, Q the unipotent radical of P, and $\Phi(Q)$ the roots of Q. We say P is **distinguished** if $\dim L/Z(G)$ equals the number of roots in $\Phi(Q)$ with P-height equal to 1.

If Q' is the derived subgroup of Q then Richardson's Theorem (iv) implies that $\dim L/Z(G) \geq \dim Q/Q'$ for all P. If P is distinguished then

$$\dim L/Z(G) = \dim Q/Q'. \qquad (*)$$

The converse holds provided $p \neq 2$ if the Dynkin diagram of G contains double bonds, and $p \neq 3$ if the Dynkin diagram of G contains triple bonds (see [2] or [1]).

The work in [2], [4] etc. takes condition $(*)$ as the definition of distinguished, but then applies this definition only with the restrictions on p just described. Thus, the definition we have given here takes the usual list of distinguished parabolics and uses this same list even when p equals 2 or 3. We refer the reader to [4] for a list of the distinguished parabolics (note however that there is a mistake in the second formula for D_n).

Throughout this paper, a **Levi subgroup** means a Levi factor of a parabolic subgroup. Let L be a Levi subgroup of G and $u \in L$ a unipotent element. Then u is **distinguished** in L if u is not contained in any proper Levi subgroup of L. If $L = G$ has trivial center this is equivalent to having $C_G(u)$ contain no non-trivial torus (see Lemma 5.1 below).

For a reductive group G let BC-pairs(G) denote the pairs (L, P) where L is a Levi subgroup and P is a distinguished parabolic subgroup of L. Let ψ (or ψ_G) denote the map from such pairs to unipotent classes in G obtained by extending the Richardson class of P (in L) to a G-class.

Theorem 3 (Bala-Carter [2], Pommerening [13]). *Let G be a simple algebraic group and let $\psi = \psi_G$ be as just defined. The following hold:*

(i) *If X is a Levi subgroup the following diagram commutes:*

$$\begin{array}{ccc} X\textit{-classes in } \text{BC-pairs}(X) & \xrightarrow{\psi_X} & \textit{unipotent classes in } X \\ \downarrow & \circ & \downarrow \\ G\textit{-classes in } \text{BC-pairs}(G) & \xrightarrow{\psi} & \textit{uniptotent classes in } G \end{array}$$

where the vertical maps extend an X-class to the corresponding G-class.

(ii) *Let $\psi(L, P) = C$ and $u \in C \cap L$. Then u is distinguished in L.*

(iii) *The map ψ is injective. It is a bijection except in the following cases: $G \in \{B_n, C_n, D_n\}$ and $p = 2$; (G, p) is one of $(E_7, 2)$, $(E_8, 2)$, $(E_8, 3)$, $(F_4, 2)$ or $(G_2, 3)$ in which cases there are 1, 4, 1, 4 and 1 extra classes respectively.*

Remarks 1.2. Although part (i) is obvious, we state it here to bring attention to some of the following points. (a) Part (i) makes the Bala-Carter Theorem more useful than Jordan blocks for comparing unipotent classes in X and unipotent classes in G. For example, let $G = E_6$, $X = D_5T_1$ and

let C be the unipotent class of X which has two Jordan blocks of size 5 in the natural module for SO_{10}. The Jordan blocks do not make it clear which class C corresponds to in G. However, the Bala-Carter label for C is A_4 (i.e. C is represented by a regular element of a Levi subgroup of type A_4) both as a class of X and when it is extended to a class of G. (b) If X is a maximal rank reductive subgroup which is not a Levi subgroup, one may often still obtain a commutative diagram similar to that in part (i). For instance let $G = E_6$, $X = A_2A_2A_2$ and $(L, P) \in$ BC-pairs(X) where L is a proper Levi subgroup of X. Then $(L, P) \in$ BC-pairs(G) and the same result is obtained if one first extends (L, P) to a G-class and then takes the unipotent G-class, or if one first takes the unipotent X-class and then extends this to a G-class. (c) Parts (i) and (iii) show that in most cases the intersection of a unipotent G-class with a Levi subgroup forms a single unipotent class for the Levi subgroup. If this is not the case then $G = E_r$, L is of type D_n and the unipotent class is of type A_{n-1}. (d) Part (iii) is a stronger version of the Bala-Carter-Pommerening Theorem than usually appears (as in the references above or [4], [8]), although this version seems to be known or assumed by specialists in the field (see, for example, [10]). In addition, the proof given in this paper (see Proof 5.4) uses the classification of unipotent classes for each simple algebraic group whereas the standard proof (as in [4]) constructs the inverse of ψ (at the level of the Lie algebra) and is independent of these classifications.

2 Recollections and Conventions

All algebraic groups in this paper are affine and defined over a fixed algebraically closed field of characteristic $p \geq 0$.

The groups SO_{2n}, SO_{2n+1} and Sp_{2n} are defined in terms of a bilinear form and a quadratic form which we will usually denote by β and φ respectively. Let V be the natural module for one of these groups. A subspace W is **totally singular** if $\varphi\big|_W$ is identically zero (which implies that $\beta\big|_{W\times W}$ also equals zero); it is **nonsingular** if $\beta\big|_W$ has trivial radical. If $G = \mathrm{GL}_n$ we consider each subspace of its natural module to be totally singular. If W is a nonsingular subspace then $\mathrm{Cl}(W)$ denotes the classical group of the same type as G defined on W.

Let G be a classical group with natural module V. A **flag** is a sequence of nested subspaces. Let f be the flag $W_0 \leq W_1 \leq \cdots \leq W_\ell = V$. Then f has length ℓ and is **totally singular** if for each i either W_i is totally singular or $W_i = W^\perp$ for some totally singular subspace $W \leq V$ (if V is

nonsingular this is equivalent to requiring that either W_i or $W_i^\perp$ be totally singular). A subgroup of G is parabolic if and only if it is the stabilizer of a totally singular flag. Let $L = \mathrm{GL}_{n_1} \dots \mathrm{GL}_{n_s} \mathrm{Cl}_m$ be the Levi factor of a parabolic subgroup P. We say a flag f of length ℓ is a **natural flag** for P if the following hold: f is totally singular, P is the stabilizer of f, $\ell = s$ if $G = \mathrm{GL}_n$, $\ell = 2s$ if $m = 0$ and $G \in \{\mathrm{SO}_{2n}, \mathrm{Sp}_{2n}\}$, and $\ell = 2s+1$ if $m \geq 1$ or $G = \mathrm{SO}_{2n+1}$. The unipotent radical of a parabolic equals the set of elements which act trivially upon each factor in a natural flag.

In the classical groups the unipotent classes are described using partitions. We will mention only a few facts here and refer the reader to [4] or [5] for more complete information. Let G be one of GL_n, O_n, SO_n, Sp_n, let C be a unipotent class of G and let λ be the partition of n consisting of the Jordan block sizes of C.

The **parity conditions** on λ refer to the following requirements: if $G \in \{\mathrm{O}_n, \mathrm{SO}_n\}$ and $p \neq 2$ then each even part of λ must have even multiplicity; if $G = \mathrm{Sp}_n$ or $G \in \{\mathrm{O}_n, \mathrm{SO}_n\}$ and $p = 2$ then each odd part of λ must have even multiplicity; if $G = \mathrm{SO}_n$ with n even then λ must have an even number of parts.

If $G \in \{\mathrm{O}_n, \mathrm{Sp}_n\}$ and λ has no even parts with even multiplicity then all unipotent elements with Jordan blocks equal to λ form a single G-class. This is generally not the case if $G \neq \mathrm{GL}_n$ and $p = 2$.

If $u \in C$ we say a part x of λ is nonsingular if there exists a Jordan chain of u (i.e. a sequence of vectors $(v_i)_{i=0}^{x}$ such that $v_0 = 0$ and $(u-1)v_i = v_{i-1}$ for all $i \geq 1$) which generates an x-dimensional nonsingular subspace.

Lemma 2.1 ([5]). *Let G be O_n, SO_n or Sp_n, with natural module V, bilinear form β, $u \in G$ a unipotent element, λ the Jordan blocks of u, x a part of λ and $v_1, \dots, v_x$ a Jordan chain of u.*

- (i) *The subspace $\langle v_1, \dots, v_x \rangle$ is nonsingular if and only if $\beta(v_i, v_j) \neq 0$ for some or, equivalently, for all $i, j > 0$ with $i + j = x + 1$. If V is nonsingular and x has multiplicity 1 then the subspace $\langle v_1, \dots, v_x \rangle$ is nonsingular.*
- (ii) *If G equals O_n or SO_n and $p \neq 2$ then x is nonsingular if and only if x is odd. If $G = \mathrm{Sp}_n$ and $p \neq 2$ then x is nonsingular if and only if x is even. In any case, if $x \neq 1$ and the multiplicity of x is odd then x is nonsingular.*

Remarks 2.2. (a) A stronger statement than given here is possible. In particular, keeping track of information about singularity of partitions is enough to parameterize the unipotent classes of O_n and Sp_n in characteristic

2. (b) Spaltenstein [17, I.2.8] gives the following expression (using different terminology) for the dimension of the centralizer of a unipotent element. Suppose $p = 2$ and G equals Sp_n or SO_n with n even.. Let $u \in G$ be a unipotent element and let $u_{\mathbb{C}} \in \mathrm{Sp}_n(\mathbb{C})$ be a unipotent element with the same Jordan blocks as u. Then $\dim C_G(u)$ equals $\dim C_{\mathrm{Sp}_n(\mathbb{C})}(u_{\mathbb{C}})$ plus the number of even, singular parts of λ.

If G is a reductive (not necessarily connected) group, a unipotent element is **regular** if the dimension of its centralizer equals the rank of G. If G is connected then the regular elements form a single unipotent class (see [18] or [4]), which is the Richardson class of the Borel subgroups. If G is not connected then the number of regular classes is at most the number of connected components (see [17] for more on this and the connection with Richardson classes).

Lemma 2.3. *Let G be one of GL_n, SO_{2n+1}, SO_{2n}, Sp_{2n} or $\varnothing_{2n}$ and exclude the case SO_{2n+1} with $p = 2$. Let λ be the Jordan blocks of a regular unipotent class. If $G = \mathrm{GL}_n$ then $\lambda = n$. If $G = \mathrm{SO}_{2n+1}$ then $\lambda = 2n+1$. If $G = \mathrm{SO}_{2n}$ then λ equals $(2n-1, 1)$ or $(2n-2, 2)$ according as $p \neq 2$ or $p = 2$ respectively. If $G = \mathrm{Sp}_{2n}$ then $\lambda = 2n$. If $G = \varnothing_{2n}$ and $p = 2$ then there are two regular unipotent classes and these have Jordan blocks of sizes $2n$ and $(2n-2, 2)$. In all cases all parts of λ are nonsingular.*

Proof. This follows from an easy dimension calculation. □

3 Proof of Theorems 1 and 2

For this section we use the following notation and assumptions (with three explicit exceptions marked by the phrase "Contrary to our usual assumptions ..."). We assume throughout that G is one of GL_n, SO_{2n+1}, SO_{2n}, Sp_{2n} and exclude the case SO_{2n+1} when $p = 2$. Let V be the natural module for G and β the bilinear form on V if $G \neq \mathrm{GL}_n$. Let P be a proper parabolic subgroup (we allow $P = G$ in the statement of Theorem 1, but if this holds there is nothing to prove). Let Q be the unipotent radical of P and $f = (0 = W_0 < \cdots < W_\ell = V)$ a natural flag. Let $\Lambda = (n_1, \ldots, n_s) \oplus m = (1^{c(1)}, 2^{c(2)}, \ldots) \oplus m$ be the Levi partition of P.

For any $g \in Q$ we let $\lambda(g) = (\lambda_1(g), \lambda_2(g), \ldots)$ be the partition of Jordan blocks of g. We fix $u \in Q$ which represents the Richardson orbit in Q. We fix $\lambda = (\lambda_1, \lambda_2, \ldots) = \lambda(u)$ and $\mu = \psi(\Lambda)^*$. We wish to prove that $\lambda = \mu$.

Essentially the proof of Theorem 1 is inductive. We will produce the largest one or two Jordan blocks of λ and show that they equal the largest

one or two parts of μ and that they generate a nonsingular subspace V_r. We will then look at the action of u on V/V_r and induct. See Remark 3.3 for the main steps in this proof.

Theorem 2 is proven in Corollary 3.7.

Lemma 3.1. *Let the notation be as described above.*

(i) *Contrary to our usual assumptions, let $G \leq \mathrm{GL}(V)$ be any algebraic group, let C_λ and C_μ be any two unipotent classes of G with Jordan blocks given by the partitions λ and μ respectively. If $C_\lambda \leq C_\mu$ then $\lambda \leq \mu$.*

(ii) *Let $g \in Q$ and let V_r be a subspace formed by r Jordan blocks of g. Then $\dim V_r \leq \sum_{i=1}^{\ell} \min\{r, \dim W_i/W_{i-1}\}$ with equality holding if and only if $\dim V_r \cap W_j = \sum_{i=1}^{j} \min\{r, \dim W_i/W_{i-1}\}$ for all $j \geq 1$. In particular, for all $r \geq 1$ we have $\sum_{i=1}^{r} \lambda_i \leq \sum_{i=1}^{\ell} \min\{r, \dim W_i/W_{i-1}\}$.*

(iii) *Let $G \in \{\mathrm{SO}_{2n+1}, \mathrm{SO}_{2n}, \mathrm{Sp}_{2n}\}$. If $G = \mathrm{SO}_{2n+1}$ let $r = 1$ and otherwise let $r = 2$. If there exists $g \in Q$ with $(\lambda_1(g), \ldots, \lambda_r(g)) = (\lambda_1, \ldots, \lambda_r)$ such that $\lambda_1(g), \ldots, \lambda_r(g)$ are nonsingular as Jordan blocks of g then $\lambda_1, \ldots, \lambda_r$ are nonsingular as Jordan blocks of u.*

Sketch of proof. Part (i). Let U be the variety of all unipotent elements in G. It is easy to show that for each $j, b \geq 0$ the subset $\{g \in U \mid \dim\ker(g-1)^j \geq b\}$ is closed in U. (One way to prove this is to use elementary characterizations of rank in terms of determinants of minors of a matrix. Another way is to use the upper semi-continuity of dimension applied to the endomorphism of $U \times V$ given by $(g, v) \mapsto (g, (g-1)^j v)$, see [17, III.8.1].) Let $u_\lambda \in C_\lambda$ and $u_\mu \in C_\mu$. Since $C_\lambda \subseteq \overline{C_\mu}$ we have that u_λ is contained in any G-invariant, closed subset of U that contains u_μ. Thus, for each j, one has $u_\lambda \in \{g \in U \mid \dim\ker(g-1)^j \geq \dim\ker(u_\mu - 1)^j\}$. Finally, note that $\dim\ker(u_\lambda - 1)^j = \sum_{i=1}^{j} \lambda_i^*$ and $\dim\ker(u_\mu - 1)^j = \sum_{i=1}^{j} \mu_i^*$.

Part (ii) is elementary linear algebra and induction together with the fact that g acts trivially upon each factor W_i/W_{i-1}.

Part (iii) uses the following facts. Every P-invariant, nonempty, open subset of Q contains u. Let X be any subset of V and for $g \in Q$ define a subspace $V_g := \langle (g-1)^i v \mid i \geq 1, v \in X \rangle \leq V$. Then the set of $g \in Q$ such that V_g is a nonsingular subspace is an open set. (Note that one can express the fact that V_g is nonsingular via a determinant being nonzero.) □

Lemma 3.2. *If $G \in \{\mathrm{GL}_n, \mathrm{SO}_{2n+1}\}$ let $r = 1$, otherwise let $r = 2$. If the following hypotheses hold then $\lambda = \mu$.*

(i) $(\lambda_1, \ldots, \lambda_r) \leq (\mu_1, \ldots, \mu_r)$,

(ii) $\sum_{i=1}^{\ell} \min\{r, \dim W_i/W_{i-1}\} = \mu_1 + \cdots + \mu_r$,
(iii) *there exists* $g \in Q$ *with* $(\lambda_1(g), \ldots, \lambda_r(g)) = (\mu_1, \ldots, \mu_r)$ *and if* G *is orthogonal or symplectic* $\lambda_1(g), \ldots, \lambda_r(g)$ *are nonsingular as Jordan block sizes of* g.

Remark 3.3. The previous lemma abstracts the inductive step in showing $\lambda = \mu$. Essentially one can view Lemma 2.3 and Lemma 3.9 as the base cases. Lemma 3.4 finishes the proof for the case $G = \mathrm{GL}_n$. For the remaining groups Lemma 3.8 establishes part (i) and (ii) and Lemma 3.9 establishes part (iii).

Proof. If G has rank 1 then P is a Borel subgroup and we are done by Lemma 2.3. We assume now that Theorem 1 is true for classical groups with natural module V' where $\dim V' < \dim V$.

Combining hypothesis (i) and Lemma 3.1(i) gives that $(\lambda_1(g), \ldots, \lambda_r(g)) \leq (\lambda_1, \ldots, \lambda_r) \leq (\mu_1, \ldots, \mu_r)$, whence we have equality by hypothesis (iii). Let V_r be the space generated by r Jordan chains of u, of lengths $(\lambda_1, \ldots, \lambda_r) = (\mu_1, \ldots, \mu_r)$. If G is symplectic or orthogonal we apply Lemma 3.1(iii) and assume that V_r is nonsingular. By hypothesis (ii) we have $\dim V_r = \lambda_1 + \cdots + \lambda_r = \mu_1 + \cdots + \mu_r = \sum_{i=1}^{\ell} \min\{r, \dim W_i/W_{i-1}\}$.

The inductive step will proceed as follows. Let $X = V_r$. We will produce a u-stable decomposition $V = X \oplus Y$. We will show that f induces flags in X and Y which we will denote by $f \cap X$ and $f \cap Y$ such that W_i is the direct sum of corresponding terms in $f \cap X$ and $f \cap Y$. We will then calculate the Jordan blocks of the Richardson classes in $\mathrm{Cl}(X)$ and $\mathrm{Cl}(Y)$ (these are the classical groups defined on X and Y) associated with $f \cap X$ and $f \cap Y$ and show that these equal $\lambda(u\big|_X)$ and $\lambda(u\big|_Y)$. The Jordan blocks of $u\big|_X$ are $(\lambda_1, \ldots, \lambda_r) = (\mu_1, \ldots, \mu_r)$ by construction (since $X = V_r$), and the blocks of $u\big|_Y$ will be found by induction.

Let $f \cap X$ denote the flag in $X = V_r$ with terms given by $X_i := X \cap W_i$ for $1 \leq i \leq \ell$. We will construct below a space Y and a flag $f \cap Y$ with terms Y_i such that $W_i = X_i \oplus Y_i$ for $1 \leq i \leq \ell$. When G equals Sp_n or SO_n the flags will be totally singular and in all cases u will act trivially upon the factors in each flag.

Let $G = \mathrm{GL}_n$. Let Y_1 be any direct complement of X_1 in W_1; this is u-stable since u acts as 1 on W_1. Let $i \geq 2$ and suppose Y_{i-1} has been constructed such that $W_{i-1} = X_{i-1} \oplus Y_{i-1}$. Using the fact that $X_{i-1} \cap Y_{i-1} = \{0\}$ it is easy to show that $\ker(u-1)\big|_{X_i} \cap \ker(u-1)\big|_{Y_{i-1}} = \{0\}$. Then we may choose a direct complement Z and a basis $v_1, v_2, \ldots,$ of another direct

complement as indicated:

$$\begin{aligned}\ker(u-1)\big|_{W_i} &= \ker(u-1)\big|_{X_i} \oplus \ker(u-1)\big|_{Y_{i-1}} \oplus Z\\ Y_{i-1}\cap(u-1)W_i &= (u-1)Y_{i-1}\oplus\langle v_1, v_2, \dots\rangle.\end{aligned}$$

For each v_j fix $\hat{v}_j \in W_i$, a pre-image under $u-1$. Let $Y_i = Y_{i-1}\oplus\langle\hat{v}_1, \hat{v}_2, \dots\rangle\oplus Z$. To check that this sum is direct, write 0 as the sum of an element in each term, then apply $u-1$ and use the definitions. It is now relatively easy to show that $W_i = X_i \oplus Y_i$. Finally, we take $Y = Y_\ell$.

If $G \neq \mathrm{GL}_n$ let $Y = V_r^\perp$ and $Y_i = Y \cap W_i$ for $1 \le i \le \ell$. The dimension of X_i can be calculated using Lemma 3.1(ii) and $\dim Y_i$ is given by $\dim W_i + \dim(W_i^\perp \cap X) - \dim X$. Using dimension calculations one may show that $W_i = X_i \oplus Y_i$ for $1 \le i \le \ell$ and that the flags are totally singular.

For $J \in \{X, Y\}$ let $\mathrm{Cl}(J)$ be the classical group on J, let P_J be the parabolic in $\mathrm{Cl}(J)$ corresponding to the flag $f\cap J$ and let Q_J be the unipotent radical of P_J. We may identify Q_XQ_Y as a subgroup of Q. Let C denote the Richardson orbit in Q and note that $C\cap(Q_XQ_Y)$ is an open subset of Q_XQ_Y which is also dense as it contains u. Then C contains the Richardson orbits in Q_X and Q_Y. Let $u' \in C\cap(Q_XQ_Y)$ such that $u'\big|_X$ and $u'\big|_Y$ represent the Richardson orbits in Q_X and Q_Y respectively.

We have that u and u' are conjugate whence $\lambda(u) = \lambda(u')$. We also have that $\lambda(u\big|_X) = (\lambda_1, \dots, \lambda_r) = (\mu_1, \dots, \mu_r)$ by construction. We have $\lambda(u'\big|_X) \ge \lambda(u\big|_X)$ since $u'\big|_X$ represents the Richardson orbit (and using Lemma 3.1(i)). This, together with the fact that $\lambda(u') = \lambda(u)$ implies that $\lambda(u'\big|_X) = \lambda(u\big|_X)$. Thus $\lambda(u'\big|_Y) = \lambda(u\big|_Y)$ and we may assume, for our purposes, that $u = u'$.

We have $\lambda(u\big|_X) = (\lambda_1, \dots, \lambda_r)$ and $\lambda(u\big|_Y) = (\lambda_{r+1}, \dots)$. Since $(\lambda_1, \dots, \lambda_r) = (\mu_1, \dots, \mu_r)$ it suffices to show that $(\lambda_{r+1}, \dots) = (\mu_{r+1}, \dots)$. One may calculate the parts of the Levi partition of Y using the dimensions of factors in the flag $f \cap Y$. One may verify that $\Lambda(Y) = (\max\{n_i - r, 0\} \,|\, 1 \le i \le s)$ if $m = 0$ and $\Lambda(Y) = (\max\{n_i - r, 0\} \,|\, 1 \le i \le s) \oplus (m-1)$ if $m \ge 1$ and $G \neq \mathrm{SO}_{2n+1}$ and $\Lambda(Y) = (\max\{n_i - 1, 0\} \,|\, 1 \le i \le s) \oplus m$ if $G = \mathrm{SO}_{2n+1}$ (when $G = \mathrm{SO}_{2n+1}$ then m in Λ corresponds to SO_{2m+1}, but Y is even dimensional and $m-1$ or m in $\Lambda(Y)$ corresponds to SO_{2m-2} or SO_{2m}).

By induction we may apply Theorem 1 to determine the Jordan blocks of this Levi partition $\Lambda(Y)$. By analyzing the cases in Theorem 1, one finds that they equal μ with the first r rows removed. □

Proof 3.4 (Proof of Theorem 1 when $G = \mathrm{GL}_n$). Note that $\mu_1 = \ell = s$. Using Lemma 3.1(ii) it is easy to verify hypotheses (i) and (ii) of Lemma 3.2.

It remains to prove the existence of $g \in Q$ with $\lambda_1(g) = \mu_1$. Let X_1 be a one dimensional subspace of W_1 and $Y_1 \le W_1$ such that $W_1 = X_1 \oplus Y_1$. For $i \in \{2, \ldots, \ell\}$ let X_i be an i dimensional subspace of W_i such that $X_{i-1} \le X_i$ and let $Y_i \le W_i$ such that $Y_{i-1} \le Y_i$ and $W_i = X_i \oplus Y_i$. Define $f \cap X$ to be the flag in $X = X_\ell$ with terms given by the X_i and $f \cap Y$ to be the flag in $Y = Y_\ell$ with terms given by the Y_i. Let P_X be the parabolic in $\mathrm{GL}(X)$ of $f \cap X$, let Q_X be the unipotent radical of P_X and identify Q_X as a subgroup of Q. Then P_X is a Borel subgroup of $\mathrm{GL}(X)$, whence there exists an element g in Q_X which has one block of size $\ell = \mu_1$ by Lemma 2.3. □

Corollary 3.5. *Every unipotent class in* GL_n *is a Richardson class.*

This is also proven in [17, II.5.14] and in [8, 5.5].

Proof. If a unipotent class has Jordan blocks given by the partition ν then it is the Richardson class of any parabolic with Levi partition equal to ν^*. □

For the next result we introduce some notation. Let $H \le J$ be algebraic groups and let H act upon J via conjugation. Given a subset $O \subseteq J$ we denote by $\overline{O}$ the closure taken within J and by O^J the subset $\bigcup_{g \in J} O^g = \{gxg^{-1} \mid g \in J, x \in O\}$.

Lemma 3.6. *Let* $H \le J$ *be algebraic groups and use the notation described above. Let* O_1 *and* O_2 *be two* H*-classes in* H*.*

(i) *If* $O_1 \subseteq \overline{O_2}$ *then* $O_1^J \subseteq \overline{O_2^J}$*.*
(ii) *If* H *has a dense orbit in* $H \cap \overline{O_2^J}$*, has a single orbit in* $H \cap O_2^J$ *(i.e.* $H \cap O_2^J = O_2$*), and* O_1^J *is a subset of* $\overline{O_2^J}$ *then* $O_1 \subseteq \overline{O_2}$*.*
(iii) *If* H *has a single orbit in* O_1^J *(i.e.* $O_1^J = O_1$*), has finitely many orbits in* O_2^J*, and* O_1^J *is a subset of* $\overline{O_2^J}$ *then* $O_1 \subseteq \overline{O_2}$*.*

We refer to conditions (ii) and (iii) as "descending from J to H".

Proof. Part (i) We have:

$$O_1^J \subseteq \left(\overline{O_2}\right)^J = \bigcup_{g \in J} \left(\overline{O_2}\right)^g = \bigcup_{g \in J} \overline{O_2^g} \subseteq \overline{\bigcup_{g \in J} O_2^g} = \overline{O_2^J}.$$

Part (ii) We claim that:

$$O_1 \subseteq H \cap O_1^J \subseteq H \cap \overline{O_2^J} = \overline{O_2}.$$

The final equality is the one to be proved. Let C denote a dense orbit of H in $H \cap \overline{O_2^J}$. Then $C \subseteq \overline{O_2^J}$ whence $C^J \subseteq \overline{O_2^J}$ by part (i). On the

other hand, $O_2 \subseteq \overline{C}$ whence $O_2^J \subseteq \overline{C^J}$ by (i). Thus $O_2^J = C^J$ whence $C \subseteq H \cap C^J = H \cap O_2^J = O_2$ and $C = O_2$.

Part (iii). Denote the H-orbits in O_2^J by $O_2 = O_2^{g_1}, O_2^{g_2}, \ldots, O_2^{g_b}$ where $g_1 = 1$ and $g_i \in J - H$ for $i > 1$. We have: $O_1 \subseteq \overline{O_2^J} = \overline{O_2^{g_1}} \cup \cdots \cup \overline{O_2^{g_b}}$ which implies that $O_1 \subseteq \overline{O_2^{g_j}}$ for some j. Then $O_1 = O_1^{g_j{}^{-1}} \subseteq \left(\overline{O_2^{g_j}}\right)^{g_j{}^{-1}} = \overline{O_2}$.

□

Corollary 3.7. *Contrary to our usual assumptions, let $G \leq \mathrm{GL}_n$ be an algebraic group, let C_λ and C_μ be two unipotent classes with Jordan blocks given by the partitions λ and μ.*

(i) *Suppose that all the unipotent elements in G with Jordan blocks equal to μ form a single conjugacy class. Then $\lambda < \mu$ if and only if $C_\lambda < C_\mu$.*
(ii) *If $G = \mathrm{GL}_n$, or $G \in \{\O_n, \mathrm{Sp}_n\}$ and $p \neq 2$, or $G \in \{\O_n, \mathrm{Sp}_n\}$ and μ has no even parts with even multiplicity, or $G = \mathrm{SO}_n$ with n even and $p \neq 2$, then $\lambda < \mu$ if and only if $C_\lambda < C_\mu$.*

Proof. Part (i). By Lemma 3.1(i) we have that $C_\lambda < C_\mu$ implies $\lambda < \mu$. We prove the converse first for $G = \mathrm{GL}_n$.

Step 1: Since $\lambda \leq \mu$ we may fix a sequence of partitions $\lambda = \lambda^{(0)} < \lambda^{(1)} < \cdots < \lambda^{(r)} = \mu$ such that for each i we have that $\lambda^{(i)}$ and $\lambda^{(i+1)}$ differ in exactly two places, i.e. there exist exactly two indices j such that $\lambda_j^{(i)} \neq \lambda_j^{(i+1)}$ (see [9, p23]). Then by transitivity it suffices to prove that $\lambda < \mu \Rightarrow C_\lambda < C_\mu$ when λ and μ differ in exactly two places, which we now assume.

Step 2: Since λ and μ differ in exactly two places, we may find a subgroup $\mathrm{GL}_{n_1}\mathrm{GL}_{n_2}$ of G, a unipotent GL_{n_1}-class C, two unipotent GL_{n_2}-classes C_1 and C_2 with C_λ and C_μ the extensions to G of CC_1 and CC_2 respectively (i.e. the classes C_1 and C_2 correspond to the two parts where λ and μ differ). By Lemma 3.6(i) it suffices to show that $C_1 \leq C_2$ (for then $CC_1 \leq CC_2$ and $C_\lambda = (CC_1)^G \leq (CC_2)^G = C_\mu$).

Step 3. It suffices now to prove the result under the assumption that λ is a two part partition (whence μ has one or two parts). Then the difference between μ^* and λ^* is that λ^* has one extra 2 and two fewer 1's. Let $g \in C_\lambda$. By Corollary 3.5, we may find flags $f_\lambda : 0 < W_2 < W_3 < \ldots$ and $f_\mu : 0 < W_1 < W_2 < W_3 < \ldots$ such that f_λ and f_μ have corresponding Levi partitions of λ^* and μ^*, f_λ and f_μ are identical to the right of W_3, and g represents the Richardson orbit corresponding to f_λ (in particular g acts trivially upon each factor in f_λ). Then g is in the unipotent radical associated with f_μ, which in turn is contained in $\overline{C_\mu}$. Whence, $C_\lambda \subseteq \overline{C_\mu}$.

Now (i) is proven for $G = \mathrm{GL}_n$. If $G < \mathrm{GL}_n$ we may descend to G via part (ii) of the previous lemma; i.e., apply Lemma 3.6(ii) with $H = G$ and $J = \mathrm{GL}_n$ to get $\lambda < \mu$ implies $C_\lambda < C_\mu$.

Part (ii). This is immediate from part (i) (and the comments in Section 2), unless $G = \mathrm{SO}_n$ with n even, $p \neq 2$. However, part (i) holds for $\emptyset_n$ and one may descend to SO_n by applying Lemma 3.6(iii) with $H = \mathrm{SO}_n$ and $J = \emptyset_n$. ☐

The following lemma establishes Lemma 3.2(i),(ii) for the cases where $G \neq \mathrm{GL}_n$. It will also be used in Lemma 3.9 to establish Lemma 3.2(iii).

Lemma 3.8. *If $G = \mathrm{SO}_{2n+1}$ let $r = 1$ and if $G \in \{\mathrm{Sp}_{2n}, \mathrm{SO}_{2n}\}$ let $r = 2$. Recall that ℓ is the number of terms in the natural flag and that the Levi partition is $\Lambda = (1^{c(1)}, 2^{c(2)}, \dots) \oplus m$. Then $(\mu_1, \dots, \mu_r)$ are listed below. Furthermore, $(\lambda_1, \dots, \lambda_r) \leq (\mu_1, \dots, \mu_r)$ and $\sum_{i=1}^{\ell} \min\{r, \dim W_i/W_{i-1}\} = \mu_1 + \cdots + \mu_r$.*

- (i) *$G = \mathrm{SO}_{2n+1}$ and $p \neq 2$. We have $\mu_1 = \ell$.*
- (ii) *$G = \mathrm{SO}_{2n}$ and $p \neq 2$. If $m = 0$ and $c(1) \geq 1$ then $(\mu_1, \mu_2) = (\ell - 1, \ell - 2c(1) + 1)$. Otherwise we have $(\mu_1, \mu_2) = (\ell, \ell - 2c(1))$.*
- (iii) *$G = \mathrm{SO}_{2n}$ and $p = 2$. If $m = 0$ and $c(1) \geq 2$ then $(\mu_1, \mu_2) = (\ell - 2, \ell - 2c(1) + 2)$. If $m = 0$, $c(1) = 1$, or $m \geq 1$, $c(1) \geq 1$ then $(\mu_1, \mu_2) = (\ell - 1, \ell - 2c(1) + 1)$. If $c(1) = 0$ then $(\mu_1, \mu_2) = (\ell, \ell - 2c(1))$.*
- (iv) *$G = \mathrm{Sp}_{2n}$. If $m \geq 1$ and $c(1) \geq 1$ then $(\mu_1, \mu_2) = (\ell - 1, \ell - 2c(1) + 1)$. Otherwise $(\mu_1, \mu_2) = (\ell, \ell - 2c(1))$.*

Proof. Recall that $\psi(\Lambda)$ is defined in table 1 and that μ equals $\psi(\Lambda)^*$, the dual of $\psi(\Lambda)$. Thus μ_1 equals the number of parts in $\psi(\Lambda)$ and μ_2 equals the number of parts in $\psi(\Lambda)$ which are greater than or equal to 2. It is easy to verify the stated formulas for μ_1 and μ_2.

Since $\sum_{i=1}^{\ell} \min\{1, \dim W_i/W_{i-1}\} = \ell$ and $\sum_{i=1}^{\ell} \min\{2, \dim W_i/W_{i-1}\} = 2\ell - 2c(1)$ we conclude that $\sum_{i=1}^{\ell} \min\{r, \dim W_i/W_{i-1}\} = \mu_1 + \cdots + \mu_r$ and that $(\lambda_1, \lambda_2) \leq (\ell, \ell - 2c(1))$ (by Lemma 3.1(ii)). This gives the desired upper bounds on λ in case (i) or when $(\mu_1, \mu_2) = (\ell, \ell - 2c(1))$.

For all the remaining cases we start by proving that one cannot have $\lambda_1 = \ell$, whence $(\lambda_1, \lambda_2) \leq (\ell - 1, \ell - 2c(1) + 1)$. First we assume $G = \mathrm{SO}_{2n}$, $m = 0$ (whence $\ell = 2s$) and $c(1) \geq 1$. If $\lambda_1 = \ell$ there is a Jordan chain $v_1,\dots,v_{2s}$ of length $2s = \ell$. Since u acts trivially upon each factor in the flag, and since the Jordan chain has as many elements as there are terms in the flag, we see that $v_s \in W_s - W_{s-1}$ and $v_{s+1} \in W_{s+1} - W_s$. By Lemma 2.1(i) we have $\beta(v_s, v_{s+1}) \neq 0$. Let $\widetilde{W} = \langle v_{s+1} \rangle^\perp \cap W_s$. Then $\widetilde{W}$ is a totally

singular $(n-1)$-space, whence $\widetilde{W}^{\perp}/\widetilde{W}$ is a nonsingular 2-space. But v_s and v_{s+1} project to distinct, nontrivial elements in $\widetilde{W}^{\perp}/\widetilde{W}$ whence $u|_{\widetilde{W}^{\perp}/\widetilde{W}}$ is a nontrivial unipotent element of $\mathrm{SO}(\widetilde{W}^{\perp}/\widetilde{W}) = \mathrm{SO}(2)$, a contradiction. In all other cases where $(\mu_1, \mu_2) < (\ell, \ell - 2c(1))$ the parity conditions upon λ and the fact that $\lambda_1 + \lambda_2 \leq 2\ell - 2c(1)$ imply $\lambda_1 \neq \ell$.

It remains to show that if $G = \mathrm{SO}_{2n}$, $m = 0$ and $c(1) \geq 2$ then $(\lambda_1, \lambda_2) \leq (\ell - 2, \ell - 2c(1) + 2)$. However, we have shown already that $(\lambda_1, \lambda_2) \leq (\ell - 1, \ell - 2c(1) + 1)$ and if $\lambda_1 = \ell - 1$ this contradicts the parity conditions upon λ. □

Lemma 3.9. *With the usual notation, the following hold and, in particular, Lemma* 3.2*(iii) holds, whence Theorem* 1 *is proven.*

- (i) *Let* $G = \mathrm{SO}_{2n}$ *and* $p = 2$. (i)(a) *If* $\Lambda = (2^a)$ *then* $\lambda = (2a, 2a)$. (i)(b) *If* $\Lambda = (2^a, 1)$ *then* $\lambda = (2a + 1, 2a + 1)$. (i)(c) *If* $\Lambda = (2^a, 1^b)$ *with* $b \geq 2$ *then* $\lambda = (2a + 2b - 2, 2a + 2)$.
- (ii) *Let* $G = \mathrm{SO}_{2n}$ *and* $p \neq 2$. (ii)(a) *If* $\Lambda = (2^a)$ *then* $\lambda = (2a, 2a)$. (ii)(b) *If* $\Lambda = (2^a, 1^b)$ *with* $b \geq 1$ *then* $\lambda = (2a + 2b - 1, 2a + 1)$.
- (iii) *Let* $G = \mathrm{Sp}_{2n}$. (iii)(a) *If* $\Lambda = (2^a)$ *then* $\lambda = (2a, 2a)$. (iii)(b) *If* $\Lambda = (2^a, 1^b)$ *then* $\lambda = (2a + 2b, 2a)$. (iii)(c) *If* $\Lambda = (2^a) \oplus 1$ *then* $\lambda = (2a + 1, 2a + 1)$. (iii)(d) *If* $\Lambda = (2^a, 1^b) \oplus 1$ *with* $b \geq 1$ *then* $\lambda = (2a + 2b, 2a + 2)$.

We sketch two proofs of parts (i)-(iii). Neither proof seems entirely satisfactory as each contains a tedious verification of a rather simple fact.

Proof. Sketch of first proof of (i)-(iii). For the following statement, λ and μ need not have their usual definitions. Let λ and μ be two partitions and u_λ, u_μ two unipotent elements with Jordan blocks given by λ and μ respectively. Suppose μ has two parts. Then we claim that $\lambda < \mu$ implies $\dim C_G(u_\lambda) \geq \dim C_G(u_\mu)$ with the inequality strict provided the parts of μ are nonsingular.

Given the claim, we again let λ and μ have their usual definitions, whence $\lambda \leq \mu$ by Lemma 3.8. Let L be the Levi factor of the parabolic P under discussion. Then Richardson's Theorem (iii) and the claim show $\dim L = \dim C_G(u_\lambda) \geq \dim C_G(u_\mu)$. Using the expression for μ in the previous Lemma, one may check that $\dim C_G(u_\mu)$ equals $\dim L$ or $\dim L + 2$ with the latter only if both parts of μ are singular. We conclude that both parts of μ are nonsingular, and that we must have $\lambda = \mu$.

(We note that it would be circular to prove the claim by applying Spaltenstein's version of Theorem 2, as this is proven in [17] using the version of Theorem 1 which is found there.) To prove the claim directly one may manipulate the formulas for dimensions of centralizers, though this is somewhat tedious. In particular, let c be the multiplicity of λ_1^* in λ^*. If $\lambda_1^* = 5$, or $\lambda_1^* = 4$, $c \geq 2$, or $\lambda_i^* = 3$, $c \geq 4$ then one may show that $\sum_{i\geq 1}\left((\lambda_i^*)^2 - (\mu_i^*)^2\right) > 2\lambda_1^*$ which proves the claim (by examining the formulas for dimensions of centralizers). The remaining cases amount to direct calculations.

Sketch of second proof of (i)-(iii). There are two cases.

Case 1: μ has two equal parts. We claim that there exists $g \in Q$ with $\mu = \lambda(g)$. Given the claim, and using Lemma 3.1(i) and Lemma 3.8, we have $\mu = \lambda(g) \leq \lambda \leq \mu$.

To prove the claim let $\mu = (n, n)$ where n is the rank of G. The tedious part of the argument is verifying, inductively, that one may construct, using roots in $\Phi(Q)$, a root base of an A_{n-1} root system. Given this root base, the group generated by the maximal torus and the root groups corresponding to $\mathbb{Z}$-linear combinations of this base is isomorphic to GL_n. Let g in GL_n be a regular unipotent element written as the product of a nontrivial element in each root group corresponding to a root in this root base (see [18]). Then g is in Q and g has two blocks of size n in the natural embedding of GL_n in G.

Case 2: μ has two distinct parts. As stated in Section 2, one sees that G has a unique unipotent class C_μ with Jordan blocks given by μ. Let C_λ be the Richardson class of P. By Lemma 3.8 we have $\lambda \leq \mu$, and by Lemma 3.7 we have $C_\lambda \leq C_\mu$. One may easily show that $\dim C_\mu = \dim C_\lambda$ whence $C_\lambda = C_\mu$ and $\lambda = \mu$.

Sketch of proof of 3.2(iii). This proof parallels that given for the case $G = \mathrm{GL}_n$ (see Proof 3.4) so we will be brief. Recall that the natural flag f has terms W_i. We will produce a decomposition of f, by constructing and decomposing a flag $\widetilde{f}$ which is isomorphic to (whence conjugate to) f.

Suppose $G = \mathrm{SO}_{2n+1}$. Let $\widetilde{X} \leq V$ be a nonsingular subspace of dimension $2s+1 = \ell$. Choose a totally singular flag $0 < \widetilde{X}_1 < \cdots < \widetilde{X}_\ell = X$ where $\dim \widetilde{X}_i = i$. This flag corresponds to a Borel subgroup of $\mathrm{Cl}(\widetilde{X})$ which has Levi partition $\Lambda(\widetilde{X}) = (1^s)$. Suppose $G \in \{\mathrm{Sp}_{2n}, \mathrm{SO}_{2n}\}$. Let $\widetilde{X} \leq V$ be a nonsingular subspace of dimension $2\ell - 2c(1)$. Choose a totally singular flag $0 < \widetilde{X}_1 < \cdots < \widetilde{X}_\ell = X$ where $\dim X_j = \sum_{i=1}^{j} \min\{2, \dim W_i/W_{i-1}\}$. This flag has Levi partition $\Lambda(\widetilde{X})$ as follows: If $m = 0$ then $\Lambda(\widetilde{X}) = (2^{s-c(1)}, 1^{c(1)})$; If $m \geq 1$ and $G = \mathrm{Sp}_{2n}$ then $\Lambda(\widetilde{X}) = (2^{s-c(1)}, 1^{c(1)}) \oplus 1$; If $m \geq 1$ and

$G = \mathrm{SO}_{2n}$ then $(2^{s-c(1)}, 1^{c(1)+1})$.

In each case we define $\widetilde{Y} = \widetilde{X}^{\perp}$ and choose a totally singular flag $0 \leq \widetilde{Y}_1 \leq \cdots \leq \widetilde{Y}_\ell = \widetilde{Y}$ where $\dim Y_i = \dim W_i - \dim X_i$. Define the flag $\widetilde{f}$ to have terms $0 < \widetilde{W}_1 < \cdots < \widetilde{W}_\ell = V$ where $\widetilde{W}_i = \widetilde{X}_i \oplus \widetilde{Y}_i$. Since $\widetilde{f}$ is conjugate to f we see that a similar decomposition holds for f which we express as $V = X \oplus Y$, $f = (f \cap X) \oplus (f \cap Y)$. The Levi partitions for the flag $f \cap X$ are the Levi partitions listed for $\Lambda(\widetilde{X})$ above.

Now that f has been decomposed, let P_X be the parabolic in $\mathrm{Cl}(X)$ and let Q_X be its unipotent radical. Then we identify Q_X as a subgroup of Q. Let $g \in Q_X \leq Q$ which represents the Richardson orbit in Q_X. Then we may apply parts (i)-(iii) to calculate the Jordan blocks of g. We find that $\lambda(g) = \mu_1$ if $G = \mathrm{SO}_{2n+1}$ and $\lambda(g) = (\mu_1, \mu_2)$ if $G \in \{\mathrm{SO}_{2n}, \mathrm{Sp}_{2n}\}$ where μ_1 and μ_2 are as in Lemma 3.8. □

4 Richardson Classes of Distinguished Parabolics

Lemma 4.1. *Let G be one of GL_n, SO_{2n+1}, SO_{2n}, and Sp_{2n}. Let Ψ denote the map taking each distinguished parabolic class to the Jordan blocks of its Richardson class. Then Ψ gives a bijection with the set of partitions described in table 2.*

For $p \neq 2$, the descriptions in table 2 of the image of Ψ are stated in [2], but it is not stated there that these partitions equal the Jordan blocks of the Richardson class.

Proof. If $G = \mathrm{GL}_n$, then the only distinguished parabolic is the Borel subgroup, which corresponds to the regular class.

We give the proof for SO_{2n} and leave the other cases (which are simpler) to the reader.

Let Λ be the Levi partition of a distinguished parabolic P. Using the description of distinguished parabolics given in [4] we may write $\Lambda = (n_1, \ldots, n_s) \oplus m = (1^{c(1)}, \ldots, (2m)^{c(2m)}, (2m+1)^{c(2m+1)}) \oplus m$ where we index the n_i such that n_s is the largest n_i. If $m = 0$ then $n_s \in \{1, 2\}$; if $m \geq 1$ then $n_s \in \{2m-1, 2m\}$ and in all cases $c(i) \geq 1$ if and only if $1 \leq i \leq n_s$.

Let ψ be the map defined in Theorem 1. If $p \neq 2$ and $m = 0$ then $\psi(\Lambda) = (1^{2c(1)-2}, 2^{2c(2)+1})$. If $p \neq 2$ and $m \geq 2$ then $\psi(\Lambda) = (1^{2c(1)}, \ldots, (2m-1)^{2c(2m-1)}, (2m)^{c(2m)+1})$. If $p = 2$ and $m = 0$ then $\psi(\Lambda) = (1^{2c(1)-4}, 2^{2c(2)+2})$. If $p = 2$ and $m \geq 2$ then $\psi(\Lambda) = (1^{2c(1)-2}, 2^{2c(2)+2}, \ldots, (2m-1)^{2c(2m-1)-2}, (2m)^{2c(2m)+2})$.

Table 2: Jordan blocks of distinguished Richardson classes

Image of Ψ
GL_n The partition of n consisting of a single block
$G = \mathrm{SO}_{2n+1},\ p \neq 2$ Partitions of $2n+1$ consisting of distinct odd parts
$G = \mathrm{SO}_{2n+1},\ p = 2$ Partitions of $2n+1$ of the form $1 \oplus \lambda$ such that: each part of λ is even; the multiplicity of each part of λ is at most 2; if i is even then $\lambda_i - \lambda_{i+1} \geq 4$.
$G = \mathrm{Sp}_{2n}$ Partitions of $2n$ consisting of distinct even parts
$G = \mathrm{SO}_{2n},\ p \neq 2$ Partitions of $2n$ consisting of distinct odd parts
$G = \mathrm{SO}_{2n},\ p = 2$ Partitions λ of $2n$ such that: λ has an even number of parts; each part of λ is even; the multiplicity of each part is at most 2; if i is even with $\lambda_{i+1} \neq 0$ then $\lambda_i - \lambda_{i+1} \geq 4$.

We have that $\Psi(P)$ equals $\psi(\Lambda)^*$, the dual of $\psi(\Lambda)$. The formulas for $\psi(\Lambda)$ make it clear that Ψ is injective.

Let μ be any partition and let $m(i)$ be the multiplicity of i in the dual partition μ^*. Then μ consists of distinct parts if and only if μ^* contains each integer between 1 and its maximal part. Also, each part of μ is odd if and only if for each j we have $\sum_{i\geq j} m(i)$ is odd.

If P is given and $p \neq 2$, the comments just made about μ show that $\Psi(P) = \psi(\Lambda)^*$ satisfies the properties described in the statement of the Lemma. In other words, the image of Ψ is in the desired set.

Conversely, let $p \neq 2$ and let λ be given which satisfies the properties described in the statement of the Lemma. The comments just made about μ show that λ^* can be set equal to an expression of the form given for $\psi(\Lambda)$, and then one may solve for $c(1)$, $c(2)$, etc. In other words, Ψ is surjective.

The case for $p = 2$ may be verified similarly, however the following alternative description may make the proof easier. Let $\lambda^{(2)}$ and $\lambda^{(\neq 2)}$ be the Jordan blocks of the Richardson class of a parabolic associated with Λ when $p = 2$ and when $p \neq 2$ respectively. Let $\lambda^{(\neq 2)} = (\lambda_1, \lambda_2, \ldots, \lambda_{2\ell-1}, \lambda_{2\ell})$ where $\lambda_{2\ell-1}$ or $\lambda_{2\ell}$ is the last nonzero part of $\lambda^{(\neq 2)}$. Then $\lambda = (\lambda_1 - 1, \lambda_2 + 1, \ldots, \lambda_{2\ell-1} - 1, \lambda_{2\ell} + 1)$. This description may be verified directly from the formulas for $\psi(\Lambda)$ (Spaltenstein [17, III.7.2,III.8.2] defines a similar map for the Jordan blocks of all unipotent classes; note there is a typographical mistake in the formula for SO_{2n+1}).

Given $\lambda = \psi(\Lambda)^*$, it remains to prove that λ is nonsingular. For those cases where λ has distinct parts this follows from Lemma 2.1. In the remaining cases we have that G is orthogonal and $p = 2$. By Richardson's Theorem (iii) we know $\dim L = \dim C_G(u)$ where L is the Levi subgroup determined by Λ and u is an element of the Richardson class in G. It is now easy to finish the proof by using Spaltenstein's expression for $\dim C_G(u)$ described in Remarks 2.2. $\square$

Corollary 4.2. *Let G be a simple algebraic group and consider the map which takes each distinguished parabolic class to its Richardson class. This map is injective.*

Proof. For the classical groups this follows from the previous lemma. For the exceptional groups, we observe that no two distinct distinguished parabolics have the same dimension of Levi factor. By Theorem 1 (iii) the dimension of the Levi factor equals the dimension of the centralizer of an element in the unipotent class, whence the result follows by dimension. $\square$

5 Proof of the Bala-Carter-Pommerening Theorem

Throughout this section, G denotes a connected reductive group, unless indicated otherwise.

Lemma 5.1. (i) *Let S be a torus in G. Then $L = C_G(S)$ is a Levi subgroup.*

(ii) *If u is a unipotent element and S a maximal torus of $C_G(u)$ then u is distinguished in $L = C_G(S)$. Furthermore, any Levi subgroup in which u is distinguished is conjugate to L via an element of $C_G(u)^\circ$.*

Proof. For part (i) one may adapt [4, 5.9.2]. For part (ii) one may adapt [4, 5.9.3]. □

Corollary 5.2. *Define a map from G-classes of pairs (L, C) consisting of a Levi subgroup L of G and a distinguished unipotent L-class C to unipotent G-classes by extending C. This map gives a bijection.*

Lemma 5.3. *Let P be a distinguished parabolic of G. Let $\overline{G} = G/Z(G)$, $\overline{P} = P/Z(G)$, let $\overline{Q}$ be the unipotent radical of $\overline{P}$ and let u represent the dense orbit of $\overline{P}$ upon its unipotent radical $\overline{Q}$. Then $C_{\overline{G}}(u)^\circ = C_{\overline{P}}(u)^\circ = C_{\overline{Q}}(u)^\circ$. In particular the Richardson class of P is distinguished in G.*

Proof. It is easy to reduce to the case $Z(G) = 1$ and adapt the proof given in [4, 5.8.7]. □

Proof 5.4 (Proof of Theorem 3). Part (i). This is by definition of the map ψ.

Part (ii). We have $\psi(L, P) = C$ and $u \in C \cap L$. Let $M \leq L$ be a minimal Levi subgroup containing u. We wish to show that $L = M$. By definition, C is obtained by extending to G the Richardson class in L of P. If $v \in L$ represents this Richardson class in L then v is distinguished in L by Lemma 5.3. Since u is conjugate to v (in G) we have $\operatorname{rank} C_G(u) = \operatorname{rank} C_G(v)$. By Lemma 5.1 we have $\dim Z(M) = \operatorname{rank} C_G(u) = \operatorname{rank} C_G(v) = \dim Z(L)$ whence $L = M$.

Part (iii). Corollary 4.2 shows that ψ, restricted to those pairs where $L = G$, is injective and part (ii) shows that the image of this restriction is a subset of the distinguished classes of G. Then Corollary 5.2 shows that ψ defined on all of BC-pairs(G) is injective.

For surjectivity, we have two cases. If G is a classical group, we use the description of distinguished unipotent classes in [17, II.7.10] or [5] and apply

Lemma 4.1 to see that ψ, applied to those pairs (L, P) where $L = G$, has image equal to all the distinguished classes of G. Then Corollary 5.2 shows that ψ is surjective. If G is exceptional it is simpler to count all pairs (L, P) and compare this to the number of unipotent classes in G as found in [10], which draws on [11], [12], [15], [16], [21]. □

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